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FORMULATION OF THE COMPLETE  
EQUATIONS OF BOUNDARY LAYER  
STABILITY WITH MASS TRANSFER

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Aerodynamics Research Report No. 271

FORMULATION OF THE COMPLETE EQUATIONS OF BOUNDARY LAYER  
STABILITY WITH MASS TRANSFER

by

John O. Powers

**ABSTRACT:** The stability theory of laminar boundary layers in response to infinitesimal disturbances is re-examined for the case of a binary mixture with foreign gas injection. Because of the inherent limitations of the asymptotic stability calculation procedures, an approach was taken which utilizes the complete equation system. Such an approach is described herein and the resulting system of equations is presented in a manner suitable for numerical evaluation.

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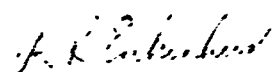
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**Formulation of the Complete Equations of Boundary Layer Stability  
with Mass Transfer**

The present analysis extends the methods of direct solution of the boundary layer disturbance equations to account for the effects of foreign gas injection on the laminar boundary layer stability. The method is based on the work of W. B. Brown and L. M. Mack, with the differences coming from the addition of a species continuity equation, diffusive flux terms in the energy equation, a modification of the form of the equation of state, and a difference in the dependency of the transport properties on the state variables.

The present report covers only the formulation of the equation system to a point where solutions may be sought by numerical methods.

**E. F. SCHREITER**  
Captain, USN  
Commander

  
**K. R. ENKENHUS**  
By direction

NOLTR 66-187

CONTENTS

	<u>Page</u>
INTRODUCTION.....	1
ANALYSIS.....	2
General Equations.....	2
Two Dimensional Parallel Flow.....	5
The Model Function Equations.....	8
Thermodynamic and Transport Properties.....	10
Reduction to Normal Form.....	14
Method of Solution.....	19
CONCLUDING REMARKS.....	24
REFERENCES.....	25

SYMBOLS

$A_1^{(j)}$	factor in the general solution of equation (98) defined in equation (129), and given by equations (130) through (137)
$a_{ij}$	complex coefficients defined by equation (60)
$B_1^{(j)}$	factor defined in equation (114) and given by equations (123) through (128)
$b_{ij}$	complex coefficients which are combinations of the $a_{ij}$ ; given by equations (103) through (113)
$C_p$	specific heat
$CP$	specific heat ratio defined in equation (23)
$c$	mass concentration
$c_j$	coefficients defined by equation (129)
$c_i$	imaginary component of disturbance velocity
$c_r$	real component of disturbance velocity
$\mathcal{D}_{ab}$	binary diffusion coefficient
$d$	molecular diameter
$e_{ij}$	strain tensor
$F(cb)$	function defined by equation (12)
$f(\eta)$	mode function for $u'$
$G$	quantity defined in equation (34)
$h$	enthalpy
$L$	quantity defined by equation (83)
$l$	length scale used in non-dimensionalizing the equation system, $= \frac{x}{\sqrt{Re_x}}$
$m$	molecular weight
$m_n(\eta)$	mode functions for the variables given in page 8, $n = 1, 2, 3, 4, 5, \text{ and } 6$
$Pr$	Prandtl number

SYMBOLS (Cont'd)

$p$	pressure
$Q$	general designation for a quantity
$R$	gas constant
$r(\eta)$	mode function for $\rho'$
$Re$	Reynolds number
$Sc$	Schmidt number
$T$	temperature
$u$	velocity parallel to plate
$u_i$ (or $u_j$ )	general designation for velocity in the $i^{th}$ (or $j^{th}$ ) direction
$V_n$	set of dependent variables defined in page 19, $n = 1, 2, 3$ , and 4
$v$	velocity normal to plate
$w(\eta)$	dimensionless velocity ratio, defined in page 8
$x_i$	Cartesian coordinate in the $i^{th}$ direction
$y$	coordinate normal to the plate
$Z_n$	set of dependent variables defined in equation (59), $n = 1, 2, 3, 4, 5, 6, 7$ , and 8
$\alpha$	wave number of the disturbance
$\beta_{ij}$	quantity defined in equation (34)
$\gamma$	ratio of specific heats
$\delta_{ij}$	Kroenicker delta
$\eta$	Blasius' similarity variable, $\frac{y}{x} \sqrt{Re_x}$
$\Omega_n$	quantities defined in equations (55) through (58), $n = 1, 2, 3$ , and 4
$\theta(\eta)$	mode function for $h'$
$\lambda$	viscosity coefficients

SYMBOLS (Cont'd)

$\lambda_j$	exponents in the particular solutions (Eq. (114)) given by equations (119) through (122)
$\mu$	coefficient of bulk viscosity
$\nu$	coefficient of kinematic viscosity
$\xi(\eta)$	mode function for $C_b'$
$\pi(\eta)$	mode function for $p'$
$\rho$	density
$\tau$	time scale used in non-dimensionalizing equation system, $\frac{l}{u_\infty}$
$\tau_{ij}$	stress tensor
$\phi(\eta)$	mode function for $v'$

Subscripts

a	refers to mean flow gas
b	refers to injected gas
$\delta$	refers to boundary layer thickness

## INTRODUCTION

The effect of foreign gas injection on the stability of the laminar boundary layer was first analyzed by Shen, reference (1), in 1957 using asymptotic procedures. He developed the "inviscid solutions" in power series of the square of the wave numbers,  $\alpha^2$ , and then demonstrated how the "viscous solutions" could be obtained by an asymptotic expansion in the inverse square root of the product of the wave number and the Reynolds number,  $(\alpha Re_\delta)^{-\frac{1}{2}}$ . By further manipulation he was able to show that the influence of foreign gas injection was evident primarily through the mean boundary layer profiles and through a simple correction factor in the well-known Dunn and Lin secular equation, reference (2). In addition, Shen developed an "inviscid criterion" for the stability of injection profiles and by applying this criterion he was able to demonstrate that the injection of a heavy molecular weight gas as a coolant might lead to boundary layer with improved stability characteristics.

The possibility of improving the stability characteristics of boundary layers by injection of heavy molecular weight gases stimulated the research of Powers, Heiche, and Shen, reference (3), who made a qualitative investigation of this phenomenon. They reformulated the asymptotic procedures of reference (1) to facilitate numerical solution of the stability equations and made a systematic investigation of the effects of varying the molecular weight and diameter of the injected gas. It was shown that, in terms of minimum critical Reynolds number, the injection of a small-diameter light-weight gas could decrease the stability by as much as an order of magnitude. In contrast, the injection of a large-diameter heavy gas could actually improve the stability of the boundary layer. Those results were for zero Mach number thermal boundary layers; however, the findings were found to apply up to low supersonic Mach numbers in a later investigation, reference (4).

Attempts to extend the results to Mach numbers higher than about 1.3 by the asymptotic approach were not possible because of an inherent limitation in the procedure. This limitation is believed to be associated with the apparent singularity in the "inviscid" equation. The singularity occurs when the velocity of the disturbance relative to the wall becomes supersonic and as a result one of the coefficients of the differential equations changes sign. This, in turn, violates one of the conditions of the Sturm-Liouville theorem which guarantees the existence of eigensolutions only under certain conditions. It has been possible to obtain solutions at high Mach numbers by changing dependent variables in a manner suggested by Lees and Lin, reference (5). Such solutions were obtained by



Reshotko, reference (6), but they were not unique eigensolutions. L. Mack, reference (7), demonstrated in fact that the multiplicity of solutions increased as the Mach number increased.

The use of asymptotic procedures at high Mach numbers would require the inclusion of many terms which had previously been neglected and as a result the numerical program would become very complex. Under these circumstances it was considered desirable to investigate the high Mach number effects of mass transfer on the boundary layer stability by developing a method of directly solving the complete linearized disturbance equations by numerical methods. The feasibility of the direct solution method has already been established by the excellent pioneering work of Drs. W. Byron Brown, reference (8), and L. Mack, reference (9). Both of these investigations have been well documented and may be considered as the foundational work upon which the present development is based. It is, therefore, the objective of the present study to extend these previous methods to accommodate the effects of foreign gas injection on the laminar boundary layer stability. The present investigation is a formulation of the equation system, and methods of obtaining numerical solutions are indicated. Numerical solutions are not included in the present report since it was considered desirable to expedite the dissemination of the formulation before complete solutions were obtained.

## ANALYSIS

### General Equations

The general equations of motion for a binary gas system in Cartesian tensor notation are used as the basic equations for the present analysis. These equations are the equations of motion, global continuity, species continuity, and energy. In the dimensional form they are the following:

$$\text{Motion: } \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = \frac{1}{\rho} \frac{\partial \tau_{ij}}{\partial x_j} \quad (1)$$

$$\text{Global Continuity: } \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j} (\rho u_j) = 0 \quad (2)$$

$$\text{Species Continuity: } \rho \left\{ \frac{\partial c_b}{\partial t} + u_j \frac{\partial c_b}{\partial x_j} \right\} = \frac{\partial}{\partial x_j} \left( \rho D_{ab} \frac{\partial c_b}{\partial x_j} \right) \quad (3)$$

$$\begin{aligned} \text{Energy: } \rho \left\{ \frac{\partial h}{\partial t} + u_i \frac{\partial h}{\partial x_i} \right\} &= \frac{\partial p}{\partial t} + u_j \frac{\partial p}{\partial x_j} + \tau_{ij} e_{ij} + \\ &+ \frac{\partial}{\partial x_j} \left\{ \frac{\mu}{Pr} \frac{\partial h}{\partial x_j} + \frac{\mu}{Sc} \left( 1 - \frac{Sc}{Pr} \right) (h_b - h_a) \frac{\partial x_b}{\partial x_j} \right\} \end{aligned} \quad (4)$$

In these equations all quantities take their conventional definitions as defined in the list of symbols. The terms which result from the inclusions of mass injection are related to the quantities:  $C_b$ , the mass concentration of the injection gas;  $\delta_{ab}$ , the binary diffusion coefficient for the diffusion of gas "b" into gas "a";  $Sc$ ,  $(\mu/\rho\theta)$ , the Schmidt number; and,  $h_a$  or  $h_b$ , the enthalpies of the components species. The stress and rate-of-strain tensors take their usual form:

$$\tau_{ij} = 2\lambda e_{ij} + \left\{ \frac{2}{3} (1-\mu) e_{kk} - p \right\} \delta_{ij} \quad (5)$$

$$\text{and } e_{ij} = \frac{1}{2} \left\{ \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right\} \quad (6)$$

where the viscosity coefficient,  $\lambda$ , is equal to three halves of the bulk viscosity coefficient.

In addition to equations (1) through (4) we must add to the basic set a form of the equation of state which is compatible with our application to the binary system. This is accomplished by using the conventional relation:

$$p = \rho R T \quad (7)$$

where for the binary mixture the assumption is made that the gas constant and the mean specific heat are functions only of the species concentrations. This leads to the relations:

$$R = R_a + C_b (R_b - R_a) \quad (8)$$

$$C_p = C_{pa} + C_b (C_{pb} - C_{pa}) \quad (9)$$

and

$$T = 1/C_p \quad (10)$$

By using equations (8), (9), and (10) in equation (7) and forming the logarithmic derivative the equation of state takes the form:

$$p'/p = p'/\bar{p} + h'/h + C_b'/F(C_b) \quad (11)$$

where

$$\frac{1}{F(C_b)} = \frac{R_b - R_a}{R_a + C_b(R_b - R_a)} + \frac{C_{pb} - C_{pa}}{C_{pa} + C_b(C_{pb} - C_{pa})} \quad (12)$$

In equation (11) the prime is used to indicate the fluctuation or disturbance part of a quantity which, when added to the time independent basic flow (indicated as barred), yields the instantaneous value of the quantity. This is written as:

$$Q(x_i, t) = \bar{Q}(x_i) + Q'(x_i, t) \quad (13)$$

(Note: Later the primes will be used to indicate derivatives with respect to the independent variable. This notational change is for convenience and will be noted when it occurs.)

Introducing the instantaneous form of the variables equation (13), into equations (1) through (4), subtracting the mean flow equations, and neglecting quadratic terms in the disturbance quantities results in the general form of the disturbance equations as follows:

Motion: 
$$\bar{p}' \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} + \bar{p} \left\{ \frac{\partial u_i'}{\partial t} + u_j' \frac{\partial \bar{u}_i}{\partial x_j} + \bar{u}_j \frac{\partial u_i'}{\partial x_j} \right\} = \frac{\partial \tau_{ij}'}{\partial x_j} \quad (14)$$

Global Continuity:  $\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j} (\bar{\rho} u_j' + \rho' \bar{u}_j) = 0$  (15)

Species Continuity:  $\bar{\rho} \left\{ \frac{\partial c_b'}{\partial t} + \bar{u}_j \frac{\partial c_b'}{\partial x_j} + u_j' \frac{\partial \bar{c}_b}{\partial x_j} \right\} + \rho' \bar{u}_j \frac{\partial \bar{c}_b}{\partial x_j}$   
 $= \frac{\partial}{\partial x_j} \left\{ \left( \frac{\bar{u}}{S_c} \right) \frac{\partial c_b'}{\partial x_j} + \left( \frac{u}{S_c} \right)' \frac{\partial \bar{c}_b}{\partial x_j} \right\}$  (16)

Energy:  $\rho' \bar{u}_j \frac{\partial \bar{h}}{\partial x_j} + \bar{\rho} \left\{ \frac{\partial h'}{\partial t} + u_j' \frac{\partial \bar{h}}{\partial x_j} + \bar{u}_j \frac{\partial h'}{\partial x_j} \right\} = \frac{\partial p'}{\partial t} +$   
 $+ \bar{u}_j \frac{\partial p'}{\partial x_j} + u_j' \frac{\partial \bar{p}}{\partial x_j} + \bar{c}_{ij} c_{ij}' + c_{ij}' \bar{c}_{ij} +$  (17)  
 $+ \frac{\partial}{\partial x_j} \left\{ \left( \frac{\bar{u}}{Pr} \right) \frac{\partial h'}{\partial x_j} + \left( \frac{u}{Pr} \right)' \frac{\partial \bar{h}}{\partial x_j} \right\} + \frac{\partial}{\partial x_j} \left\{ \frac{u}{S_c} \left( 1 - \frac{S_c}{Pr} \right) \times \right.$   
 $\times \left[ \left( \frac{C_{p_b} - C_{p_a}}{C_p} \right)' \bar{h} \frac{\partial \bar{c}_b}{\partial x_j} + \left( \frac{C_{p_b} - C_{p_a}}{C_p} \right) (\bar{h} \frac{\partial c_b'}{\partial x_j} + h' \frac{\partial \bar{c}_b}{\partial x_j}) \right] +$   
 $+ \left[ \frac{u}{S_c} \left( 1 - \frac{S_c}{Pr} \right) \right]' \left( \frac{C_{p_b} - C_{p_a}}{C_p} \right) \bar{h} \frac{\partial \bar{c}_b}{\partial x_j} \}$

#### Two Dimensional Parallel Flow

The only simplifying assumptions are now made; namely, that the mean flow is parallel and that the disturbances and the mean flow are two dimensional. The first of these assumptions implies that  $\bar{v} \ll \bar{u}$  and that derivatives of mean flow quantities with respect to  $x$  may be neglected, (i.e.,  $\frac{\partial \bar{Q}}{\partial x} \ll \frac{\partial \bar{Q}}{\partial y}$ ). The second assumption eliminates the coordinate dependence from the problem. These assumptions are

conventional in boundary layer stability and no attempt is made here to evaluate their significance since this has been the subject of previous investigations. (See, for example, reference (2).) As a result of these assumptions, equations (14) through (17) can be written as follows:

$$\begin{aligned} \text{x-momentum: } \bar{\rho} \left\{ \frac{\partial u'}{\partial t} + v' \frac{\partial \bar{u}}{\partial y} + \bar{u} \frac{\partial u'}{\partial x} \right\} = & - \frac{\partial p'}{\partial x} + \bar{\mu} \left\{ 2 \frac{\partial^2 u'}{\partial x^2} + \right. \\ & + \frac{\partial^2 u'}{\partial y^2} + \frac{\partial^2 v'}{\partial x \partial y} \left. \right\} + \frac{2}{3} (\bar{\lambda} - \bar{\mu}) \left( \frac{\partial^2 u'}{\partial x^2} + \frac{\partial^2 v'}{\partial x \partial y} \right) + \mu' \frac{\partial^2 \bar{u}}{\partial y^2} + \frac{\partial u'}{\partial y} \frac{\partial \bar{u}}{\partial y} + \\ & + \frac{\partial \bar{u}}{\partial y} \left( \frac{\partial u'}{\partial y} + \frac{\partial v'}{\partial x} \right) \end{aligned} \quad (18)$$

$$\begin{aligned} \text{y-momentum: } \bar{\rho} \left\{ \frac{\partial v'}{\partial t} + \bar{u} \frac{\partial v'}{\partial x} \right\} = & - \frac{\partial p'}{\partial y} + \bar{\mu} \left\{ 2 \frac{\partial^2 v'}{\partial y^2} + \frac{\partial^2 v'}{\partial x^2} + \right. \\ & + \frac{\partial^2 u'}{\partial x \partial y} \left. \right\} + \frac{2}{3} (\bar{\lambda} - \bar{\mu}) \left( \frac{\partial^2 u'}{\partial x \partial y} + \frac{\partial^2 v'}{\partial y^2} \right) + \frac{\partial \bar{u}}{\partial x} \frac{\partial \bar{u}}{\partial y} + 2 \frac{\partial \bar{u}}{\partial y} \frac{\partial v'}{\partial y} + \\ & + \frac{2}{3} \left\{ \frac{\partial \bar{\lambda}}{\partial y} - \frac{\partial \bar{\mu}}{\partial y} \right\} \left( \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} \right) \end{aligned} \quad (19)$$

Global Continuity:

$$\frac{\partial \rho'}{\partial t} + v' \frac{\partial \bar{\rho}}{\partial y} + \bar{\rho} \frac{\partial u'}{\partial x} + \bar{\rho} \frac{\partial v'}{\partial y} + \bar{u} \frac{\partial \rho'}{\partial x} = 0 \quad (20)$$

Species Continuity: 
$$\bar{\rho} \left\{ \frac{\partial \bar{c}_b'}{\partial t} + \bar{u} \frac{\partial \bar{c}_b'}{\partial x} + \bar{v}' \frac{\partial \bar{c}_b}{\partial y} \right\} = \frac{\bar{u}}{S_c} \left\{ \frac{\partial^2 \bar{c}_b'}{\partial x^2} + \frac{\partial^2 \bar{c}_b'}{\partial y^2} \right\} + \frac{\partial \bar{c}_b'}{\partial y} \frac{\partial}{\partial y} \left( \frac{\bar{u}}{S_c} \right) + \left( \frac{\bar{u}}{S_c} \right)' \frac{\partial^2 \bar{c}_b}{\partial y^2} + \frac{\partial \bar{c}_b}{\partial y} \frac{\partial}{\partial y} \left( \frac{\bar{u}}{S_c} \right)' \quad (21)$$

Energy:

$$\begin{aligned} \bar{\rho} \left\{ \frac{\partial h'}{\partial t} + \bar{u} \frac{\partial h'}{\partial x} + \bar{v}' \frac{\partial \bar{h}}{\partial y} \right\} &= \frac{\partial p'}{\partial t} + \bar{u} \frac{\partial p'}{\partial x} - \bar{\rho} \left\{ \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} \right\} \\ &+ \mu' \left( \frac{\partial \bar{u}}{\partial y} \right)^2 + 2 \bar{u} \frac{\partial \bar{u}}{\partial y} \left( \frac{\partial u'}{\partial y} + \frac{\partial v'}{\partial x} \right) + \left( \frac{\bar{u}}{\bar{\rho}} \right) \left( \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h'}{\partial x^2} \right) \quad (22) \\ &+ \frac{\partial}{\partial y} \left\{ \left( \frac{\bar{u}}{\bar{\rho}} \right)' \frac{\partial \bar{h}}{\partial y} \right\} + \frac{\partial h'}{\partial y} \frac{\partial}{\partial y} \left( \frac{\bar{u}}{\bar{\rho}} \right) + \left\{ c_p' \bar{h} \frac{\partial \bar{c}_b}{\partial y} + c_p \left[ \bar{h} \frac{\partial \bar{c}_b'}{\partial x} \right. \right. \\ &+ \bar{h} \frac{\partial \bar{c}_b'}{\partial y} + \bar{h}' \frac{\partial \bar{c}_b}{\partial y} \left. \right\} \left\{ \frac{\partial}{\partial y} \left( \frac{\bar{u}}{S_c} - \frac{\bar{u}}{\bar{\rho}} \right) \right\} + \left( \frac{\bar{u}}{S_c} - \frac{\bar{u}}{\bar{\rho}} \right) \left\{ \bar{h} \frac{\partial \bar{c}_b}{\partial y} \times \right. \\ &\times \left( \frac{\partial c_p'}{\partial x} + \frac{\partial c_p'}{\partial y} \right) + c_p' \left( \frac{\partial \bar{c}_b}{\partial y} \frac{\partial \bar{h}}{\partial y} + \bar{h} \frac{\partial^2 \bar{c}_b}{\partial y^2} \right) + \left[ \bar{h} \left( \frac{\partial \bar{c}_b'}{\partial x} + \frac{\partial \bar{c}_b'}{\partial y} \right) + \right. \\ &+ \bar{h}' \frac{\partial \bar{c}_b}{\partial y} \left. \right] \left( \frac{\partial \bar{c}_b}{\partial y} \right) + c_p \left[ \bar{h} \left( \frac{\partial^2 \bar{c}_b'}{\partial x^2} + 2 \frac{\partial^2 \bar{c}_b'}{\partial x \partial y} + \frac{\partial^2 \bar{c}_b'}{\partial y^2} \right) + \right. \\ &+ \bar{h}' \frac{\partial^2 \bar{c}_b}{\partial y^2} + \frac{\partial \bar{h}}{\partial y} \left( \frac{\partial \bar{c}_b'}{\partial x} + \frac{\partial \bar{c}_b'}{\partial y} \right) + \frac{\partial \bar{c}_b}{\partial y} \left( \frac{\partial h'}{\partial x} + \frac{\partial h'}{\partial y} \right) \left. \right\} + (c_p \bar{h} \frac{\partial \bar{c}_b}{\partial y}) \times \\ &\times \left[ \frac{\partial}{\partial x} \left( \frac{\bar{u}}{S_c} - \frac{\bar{u}}{\bar{\rho}} \right)' + \frac{\partial}{\partial y} \left( \frac{\bar{u}}{S_c} - \frac{\bar{u}}{S_c} \right)' \right] + \left( \frac{\bar{u}}{S_c} - \frac{\bar{u}}{\bar{\rho}} \right)' \left[ \bar{h} \frac{\partial \bar{c}_b}{\partial y} \frac{\partial c_p}{\partial y} + c_p \left( \frac{\partial \bar{c}_b}{\partial y} \frac{\partial \bar{h}}{\partial y} + \right. \right. \\ &+ \bar{h} \frac{\partial^2 \bar{c}_b}{\partial y^2} \left. \right) \left. \right]. \end{aligned}$$

where:  $C_p = \frac{C_{p0} - C_{p\infty}}{C_p}$  (23)

### The Mode Function Equations

It is next desirable to express the equations (11) and (18) through (22) in dimensionless form. This is accomplished by introducing the length scale,  $l = x/\sqrt{\text{Re}_x}$ , the time scale,  $\tau = l/\bar{u}_\infty$ , and by scaling the other variables with respect to their values at the edge of the boundary layer. The concentration of the injectant is an exception to this scaling since it is already dimensionless. We note that scaling results in the use of the Blasius variable,

$\eta = \frac{y}{x} \sqrt{\text{Re}_x}$ , to represent the y coordinate. It is further desirable to represent all of the fluctuations as harmonic functions whose amplitude is determined by a "mode" function of  $\eta$ . Using these considerations, the mean flow and fluctuation quantities are expressed as in the following table:

TABLE I

Mean Flow Variable	Fluctuation Quantity
$\bar{u} = \bar{u}_\infty w(\eta)$	$u' = \bar{u}_\infty f(\eta) e^{i\alpha(x-ct)}$
$\bar{v} = \bar{u}_\infty v(\eta)$	$v' = \bar{u}_\infty \alpha \phi(\eta) e^{i\alpha(x-ct)}$
$\bar{p} = \bar{p}_\infty p(\eta)$	$p' = \bar{p}_\infty \psi(\eta) e^{i\alpha(x-ct)}$
$\bar{\rho} = \bar{\rho}_\infty \rho$	$\rho' = \bar{\rho}_\infty \pi(\eta) e^{i\alpha(x-ct)}$
$\bar{h} = \bar{h}_\infty h(\eta)$	$h' = \bar{h}_\infty \theta(\eta) e^{i\alpha(x-ct)}$
$\bar{C}_b = C_b(\eta)$	$C_b' = S(\eta) e^{i\alpha(x-ct)}$
$\bar{\mu} = \bar{\mu}_\infty \mu(\eta)$	$\mu' = \bar{\mu}_\infty M_1(\eta) e^{i\alpha(x-ct)}$
$\bar{\lambda} = \bar{\mu}_\infty \lambda(\eta)$	$\lambda' = \bar{\mu}_\infty M_2(\eta) e^{i\alpha(x-ct)}$

TABLE I Cont'd.

$\frac{C_{p0}-C_{pa}}{C_p} = C_p(\eta)$	$\left(\frac{C_{p0}-C_{pa}}{C_p}\right)' = m_3(\eta) e^{i\alpha(x-ct)}$
$\overline{u/p_r} = (\overline{u/p_r})_\infty u/p_r(\eta)$	$(\overline{u/p_r})' = (\overline{u/p_r})_\infty m_4(\eta) e^{i\alpha(x-ct)}$
$(\overline{u/s_c} - \overline{u/p_r}) = (\overline{u/p_r})_\infty B(\eta)$	$(\overline{u/s_c} - \overline{u/p_r})' = (\overline{u/p_r})_\infty m_5(\eta) e^{i\alpha(x-ct)}$
$\overline{u/s_c} = (\overline{u/p_r})_\infty u/s_c(\eta)$	$(\overline{u/s_c})' = (\overline{u/p_r})_\infty m_6(\eta) e^{i\alpha(x-ct)}$

Using the definitions given in Table I, introducing the Reynolds number,  $R = \frac{u_\infty l}{\nu_\infty} = \sqrt{Re_x}$ , and using the global continuity equation in the energy equation gives the desired form of the mode function equation. (Note: For the following pages the primes will be used to indicate derivatives with respect to the variable  $\eta$ .)

x-momentum: 
$$\rho \{ i(\omega - c) f + \omega' \phi \} = \frac{-i\pi}{8M_\infty^2} + \frac{\mu}{\alpha R} \{ f'' + \alpha^2 (i\phi' - 2f) \} + \frac{2}{3} \frac{(\lambda - \mu)}{\alpha R} \alpha^2 (i\phi' - f) + \frac{1}{\alpha R} \{ m_1 w'' + m_1' w' + \mu' (f + i\alpha^2 \phi) \} \quad (24)$$

y-momentum: 
$$\rho \{ i(\omega - c) \phi \} = \frac{-\pi'}{8\alpha M_\infty^2} + \frac{\mu}{\alpha R} \{ 2\phi'' + if' - \alpha^2 \phi \} + \frac{2}{3} \frac{\lambda - \mu}{\alpha R} (\phi'' + if') + \frac{1}{\alpha R} \{ i m_1 w' + 2\mu' \phi' + \frac{2}{3} (\lambda' - \mu') (\phi' + if') \} \quad (25)$$



Global Continuity:

$$i(\omega - c) r + \rho' \phi + \phi (i f + \phi') = 0 \quad (26)$$

Species Continuity:

$$\rho \{ i(\omega - c) \xi + C_b' \phi \} = \frac{1}{R \alpha R} \left\{ \left( \frac{\mu}{S_c} \xi' \right)' + (m_b C_b')' - \alpha^2 \xi \right\} \quad (27)$$

Energy: 
$$\rho \{ i(\omega - c) \theta + \phi \kappa' \} = \frac{\gamma_\infty' - 1}{\gamma_\infty} \left\{ 2i(\omega - c) \pi - i(\omega - c) \left[ \frac{\theta}{h} + \frac{p}{F(C_b)} \right] + \frac{\rho'}{\rho} \phi \right\} + \frac{(\gamma_\infty - 1) M_\infty^2}{\alpha R} \left\{ \kappa, w'^2 + 2u w' (f' + i \alpha^2 \phi') \right\} + \frac{1}{R \alpha R} \left\{ \left( \frac{\mu}{R} \theta' \right)' - \frac{\mu}{R} \alpha^2 \theta + (m_4 \kappa')' + \left[ \left( \frac{\mu}{S_c} - \frac{\mu}{R} \right) C_p \kappa \xi' \right]' + \left[ \left( \frac{\mu}{S_c} - \frac{\mu}{R} \right) m_3 \kappa C_b' \right]' + \left[ \left( \frac{\mu}{S_c} - \frac{\mu}{R} \right) C_p \theta C_b' \right]' - \left( \frac{\mu}{S_c} - \frac{\mu}{R} \right) (\alpha^2 \kappa C_p \xi) + [(m_b - m_4) (\kappa C_p C_b')] \right\} \quad (28)$$

Equation of State:

$$\pi = \frac{r}{\rho} + \frac{\theta}{h} + \frac{p}{F(C_b)} \quad (29)$$

Before equations (24) through (29) can be developed further it is necessary to establish the form to be used for the thermodynamic and transport properties of the mixture.

#### Thermodynamic and Transport Properties

One of the major differences of the present development and the previous developments of Brown and Mack, references (8) and (9), is the manner in which the thermodynamic and transport properties enter the equation system. In the previous works, these properties are considered to be those of a single species and hence are only functions of temperature for given pressure. In the present investigation, since we are dealing with the injection of a foreign gas into a boundary layer, these properties are necessarily functions of both temperature and concentration. Since enthalpy is used herein in place

of temperature, the fluctuations of these properties and their  $\eta$  derivatives will be related respectively to the enthalpy and concentration fluctuations and to the enthalpy and concentration profiles. In the mode function equations, the properties appear both as  $\eta$  derivatives,  $\frac{dQ}{d\eta}$ , and as their fluctuation mode functions,  $m_1(\eta)$ . Because of the dual dependence these expressions are written as:

$$\frac{dQ}{d\eta} = \frac{dQ}{dh} \frac{dh}{d\eta} + \frac{dQ}{dC_b} \frac{dC_b}{d\eta} \quad (30)$$

and:  $m_i(\eta) = m_{ih} \theta + m_{icb} \xi \quad (31)$

where subscripts h and  $C_b$  indicate derivatives with respect to those quantities.

Further development of these expressions is dependent on the choice of thermodynamic and transport property relations. To achieve a reduction in complexity, the forms used by Korobkin, reference (10), are used. These mixture properties only appear as dimensionless ratios, with the exception of the Prandtl number and the ratio of specific heats at the edge of the boundary layer. These two then become the pure air values. The dimensionless ratios of the mixture properties are developed in terms of the temperature ratio, T, concentration,  $C_b$ , molecular weight,  $m_a$  or  $m_b$ , and molecular diameter,  $d_a$  or  $d_b$ , to be the following:

$$\mu = \sqrt{T} \left\{ \frac{\left(\frac{m_b}{m_a}\right)^{1/2} \left(\frac{d_a}{d_b}\right)^2}{1 + \beta_{ba} \left(\frac{1-C_b}{C_b}\right)} + \frac{1}{1 + \beta_{ab} \left(\frac{C_b}{1-C_b}\right)} \right\} \quad (32)$$

$$K = \sqrt{T} \left\{ \frac{\left(\frac{m_a}{m_b}\right)^{1/2} \left(\frac{d_b}{d_a}\right)^2}{1 + \beta_{ba} \left(\frac{1-C_b}{C_b}\right)} + \frac{1}{1 + \beta_{ab} \left(\frac{C_b}{1-C_b}\right)} \right\} \quad (33)$$

where  $\beta_{ij} = \left(\frac{d_i + d_j}{2d_i}\right)^2 \frac{m_i}{m_j} \left[\frac{m_i + m_j}{2m_j}\right]^{1/2} \quad (34)$

$$\mathcal{D} = T^{3/2} \quad (35)$$

In addition we write for convenience:

$$\frac{C}{4} = \frac{R_b - R_a}{R_a + C_b(R_b - R_a)} \quad (36)$$

Using these expressions in the proper combinations, the required relations in equations (30) and (31) become:

$$\mathcal{M}_{1h} = \frac{1}{2} \mu/h \quad (37)$$

$$\mathcal{M}_{1hh} = \frac{1}{4} \mu/h^2 \quad (38)$$

$$\mathcal{M}_{1c} = -\frac{C\rho\mu}{2} + \Omega_1 \quad (39)$$

$$\mathcal{M}_{1cc} = \frac{C\rho}{2} \left( \frac{3}{2} C\rho\mu - 2\Omega_1 \right) - \Omega_2 \quad (40)$$

$$\mathcal{M}_{1hc} = \frac{\mathcal{M}_{1c}}{2h} \quad (41)$$

$$\mathcal{M}_{3c} = -C\rho^2 \quad (42)$$

$$\mathcal{M}_{3cc} = 2C\rho^3 \quad (43)$$

$$\mathcal{M}_{4h} = \frac{1}{2h} \left( \frac{\mu}{\rho h} \right) \quad (44)$$

$$M_{4hh} = -\frac{1}{4h^2} \left( \frac{u}{P_r} \right) \quad (45)$$

$$M_{4c} = -\frac{3}{2} C_p \left( \frac{u}{P_r} \right) + \Omega_3 \quad (46)$$

$$M_{4cc} = 3C_p \left[ \frac{5}{4} C_p \left( \frac{u}{P_r} \right) - \Omega_3 \right] - \Omega_4 \quad (47)$$

$$M_{4hc} = M_{4c} / 2h \quad (48)$$

$$M_{6h} = \frac{1}{2h} \left( \frac{u}{S_c} \right) \quad (49)$$

$$M_{6hh} = -\frac{1}{4h^2} \left( \frac{u}{S_c} \right) \quad (50)$$

$$M_{6c} = -\left( \frac{C_p}{2} + G \right) \frac{u}{S_c} \quad (51)$$

$$M_{6cc} = \left[ \frac{3}{4} C_p^2 + C_p \cdot G + 2G^2 \right] \frac{u}{S_c} \quad (52)$$

$$M_{6hc} = -\left( \frac{C_p}{2} + G \right) M_{6h} \quad (53)$$

$$G_c = -G^2 \quad (54)$$

where:

$$\Omega_1 = \left[ \frac{C_{pa} h}{C_p} \right]^{1/2} \left\{ \left( \frac{m_a}{m_b} \right)^{1/2} \left[ \frac{d_a/d_b}{\beta_{ba} + (1-\beta_{ba})C_b} \right]^2 - \left[ \frac{1}{1 + (\beta_{ab}^3 - 1)C_b} \right]^2 \right\} \quad (55)$$

$$\Omega_2 = 2\sqrt{\frac{p_a h}{c_p}} \left\{ \sqrt{\frac{m_a}{m_b}} \frac{(1-\beta_{ba}) \left(\frac{d_a}{d_b}\right)^2}{[\beta_{ba} + (1-\beta_{ba})C_b]^3} - \frac{(\beta_{ab}-1)}{[1+(\beta_{ab}-1)C_b]^3} \right\} \quad (56)$$

$$\Omega_3 = \sqrt{\frac{c_{pa} h}{c_p^3}} \left\{ \sqrt{\frac{m_a}{m_b}} \left[ \frac{d_a/d_b}{\beta_{ba} + (1-\beta_{ba})C_b} \right]^2 - \left[ \frac{1}{1+(\beta_{ab}-1)C_b} \right]^2 \right\} \quad (57)$$

$$\Omega_4 = 2\sqrt{\frac{c_{pa} h}{c_p^3}} \left\{ \sqrt{\frac{m_a}{m_b}} \frac{(1-\beta_{ba}) \left(\frac{d_a}{d_b}\right)^2}{[\beta_{ba} + (1-\beta_{ba})C_b]^3} - \frac{(\beta_{ab}-1)}{[1+(\beta_{ab}-1)C_b]^3} \right\} \quad (58)$$

### Reduction to Normal Form

The equation system, equations (24) through (29), is reduced to normal form by the introduction of a new set of variables defined as follows:

$$z_1 = f ; z_2 = f' ; z_3 = \phi ; z_4 = \frac{\pi}{\gamma_\infty M_\infty^2} ; z_5 = \theta \quad (59)$$

$$z_6 = \theta' ; z_7 = \rho ; z_8 = \rho'$$

In this form we have a system of eight first order differential equations.

$$z_i' = \sum_{j=1}^8 a_{ij} z_j \quad (i=1, \dots, 8) \quad (60)$$

The coefficients,  $a_{ij}$ , are made up of combinations of the mean velocity, enthalpy, density, and concentration profiles, the thermodynamic and transport properties, and the four basic parameters of the problem; namely, the wave number,  $\alpha$ , the Reynolds number,  $R$ , the real component of the wave velocity,  $c_r$ , and the imaginary component of the wave velocity,  $c_i$ . Of the 64 possible complex coefficients,  $a_{ij}$ , only 36 are nonzero and they are the following:

$$a_{12} = 1 \quad (61)$$

$$a_{21} = \frac{i p \alpha R (w-c)}{\mu} + \alpha^2 \quad (62)$$

$$a_{22} = -\frac{1}{2} \frac{h'}{h} - \left( \frac{\Omega_1}{\mu} - \frac{c p}{2} \right) c_0' \quad (63)$$

$$a_{23} = \frac{\alpha R}{\mu} p w' + i \alpha \left\{ a_{22} + \frac{1}{2} \frac{p'}{p} \left( 1 + 2 \frac{1}{\mu} \right) \right\} \quad (64)$$

$$a_{24} = \frac{i \alpha R}{\mu} - \frac{\gamma_\infty M_\infty^2 \alpha^2}{3} (w-c) \left( 1 + 2 \frac{1}{\mu} \right) \quad (65)$$

$$a_{25} = \frac{\alpha^4}{3 h} (w-c) \left( 1 + 2 \frac{1}{\mu} \right) - \frac{1}{2 h} \left\{ w'' - w' \left( a_{22} + \frac{1}{h} \right) \right\} \quad (66)$$

$$a_{26} = -\frac{1}{2} \frac{w'}{h} \quad (67)$$

$$a_{27} = a_{25} + \frac{\alpha^2}{3} (w-c) \left( 1 + 2 \frac{1}{\mu} \right) \left( c p + G - \frac{1}{h} \right) \quad (68)$$

$$a_{28} = w' \left( \frac{c p}{2} - \frac{\Omega_1}{\mu} \right) \quad (69)$$

$$a_{31} = -i \quad (70)$$

$$a_{33} = -p'/p \quad (71)$$

$$a_{34} = -i \gamma_\infty M_\infty^2 (w-c) \quad (72)$$

$$a_{35} = \frac{i}{h} (w-c) \quad (73)$$

$$a_{37} = i (cp + G) (w-c) \quad (74)$$

$$a_{41} = \frac{2i}{L} \left\{ \frac{1}{3} \left( 2 + \frac{1}{\mu} \right) \left( \frac{p'}{\rho} \right) + a_{22} \right\} \quad (75)$$

$$a_{42} = -i/L \quad (76)$$

$$a_{43} = -\frac{1}{L} \left\{ \frac{2}{3} \left( 2 + \frac{1}{\mu} \right) \left[ \rho \left( \frac{p'}{\rho^2} \right)' - \left( \frac{p'}{\rho} \right) a_{22} \right] + a_{21} \right\} \quad (77)$$

$$a_{44} = \frac{2}{3} \frac{i \delta_{\infty} M_{\infty}^2}{L} \left( 2 + \frac{1}{\mu} \right) \left\{ a_{22} (w-c) - \rho \left( \frac{w-c}{\rho} \right)' \right\} \quad (78)$$

$$a_{45} = \frac{2}{3} \frac{i}{L} \left\{ \left( 2 + \frac{1}{\mu} \right) \left[ \rho \left( \frac{w-c}{\rho^2} \right)' - \frac{a_{22}}{h} (w-c) \right] + \frac{3}{4} \frac{w'}{h} \right\} \quad (79)$$

$$a_{46} = \frac{2i}{3Lh} \left( 2 + \frac{1}{\mu} \right) (w-c) \quad (80)$$

$$a_{47} = \frac{2i}{3L} \left\{ \left[ 2 + \frac{1}{\mu} \right] \left[ w' (cp + G) - (w-c) \left\{ \left( \frac{p'}{\rho} + a_{22} \right) (cp + G) + (cp^2 + G^2) \frac{p'}{\rho} \right\} \right] - \frac{3}{2} a_{28} \right\} \quad (81)$$

$$a_{48} = \frac{2i}{3L} (cp + G) \left( 2 + \frac{1}{\mu} \right) (w-c) \quad (82)$$

where:

$$(L = \frac{R}{\alpha \mu} + \frac{2}{3} i \gamma_{\infty} M_{\infty}^2 (2 + \frac{1}{\mu})(\omega - c)) \quad (83)$$

$$a_{56} = 1 \quad (84)$$

$$a_{62} = -2 P_{\infty} (\gamma_{\infty} - 1) M_{\infty}^2 \mu (\frac{P_r}{\mu}) \omega' \quad (85)$$

$$a_{63} = P_{\infty} \frac{P_r}{\mu} \left\{ \rho \alpha R \left[ h' - C_p h C_b' - \frac{\gamma_{\infty} - 1}{\delta_{\infty}} \frac{\rho'}{\rho^2} \right] - 2 i (\gamma_{\infty} - 1) M_{\infty}^2 \alpha^2 \mu \omega' \right\} \quad (86)$$

$$a_{64} = -2 i P_{\infty} M_{\infty}^2 \alpha R (\gamma_{\infty} - 1) \frac{P_r}{\mu} (\omega - c) \quad (87)$$

$$a_{65} = \frac{P_r}{\mu} \left\{ i P_{\infty} \alpha R (\omega - c) \left[ \rho + \frac{\gamma_{\infty} - 1}{\delta_{\infty} h} \right] - P_{\infty} (\gamma_{\infty} - 1) M_{\infty}^2 \times \right. \\ \left. \omega'^2 \frac{\mu}{2 h} - \frac{C_b'}{2} \frac{h'}{h} \Omega_3 - C_p C_b'' (\frac{\mu}{\delta_{\infty}}) - C_p C_b' \left[ \frac{\mu}{\delta_{\infty}} \frac{h'}{h} - \right. \right. \quad (88) \\ \left. \left. - C_b' (2 C_p \frac{\mu}{\delta_{\infty}} + G \frac{\mu}{\delta_{\infty}} + \frac{3}{2} \Omega_3) \right] \right\} + \alpha^2 - \frac{h''}{\alpha h} + \frac{1}{4} (\frac{h'}{h})^2 + \\ + \frac{3}{2} C_p (C_b'' + C_b' \frac{h'}{h} - \frac{5}{2} C_p C_b'^2)$$

$$a_{66} = C_b' \left\{ 3 C_p - \frac{P_r}{\mu} (\Omega_3 + C_p \frac{\mu}{\delta_{\infty}}) \right\} - \frac{h'}{h} \quad (89)$$

$$a_{67} = \frac{P_r}{\mu} \left\{ i P_{\infty} \alpha R (\omega - c) \left[ \frac{\gamma_{\infty} - 1}{\delta_{\infty}} (G + C_p) - \rho C_p h \right] - P_{\infty} (\gamma_{\infty} - 1) \times \right. \quad (90) \\ \left. M_{\infty}^2 \omega'^2 (\Omega_1 - \frac{C_p \mu}{2}) - \Omega_3 (h'' + \frac{h'^2}{2 h}) + C_b' h' [\Omega_4 + \right.$$



(Eq. (90) continued)

$$\begin{aligned}
 & + \frac{\rho}{2} \Omega_3 + \frac{\rho}{2} (3cp - \frac{1}{h}) \frac{u}{s_c} + C_b'^2 h [cp (\frac{3}{2} cp - \\
 & 2cp \Omega_3 + 1) + (\Omega_4 + 3cp \Omega_3 + \frac{cp^2}{2h} - cp^2 G - \\
 & \frac{cp^3}{2}) \frac{u}{s_c}] + Cp C_b'' h [Cp \frac{u}{s_c} + \Omega_3] - 3h C_b' cp^2 \Omega_3 \} + \\
 & + \frac{3}{2} Cp (h'' + \frac{Cp}{2} \frac{h'^2}{h}) + \frac{15}{2} C_b' (\frac{Cp^3}{2} h - h') + \\
 & Cp (\frac{3}{2} cp^2 - 3\Omega_3 + \alpha^2 h) + \Omega_4 + \frac{5}{2} Cp^2 [\frac{3}{2} (\frac{u}{h} - \\
 & \frac{u}{s_c}) - C_b'' h]
 \end{aligned}$$

$$\begin{aligned}
 a_{68} = \frac{\rho}{h} \{ & Cp C_b' h [\Omega_3 + 2cp \frac{u}{s_c}] - h' [\Omega_3 + Cp \frac{u}{s_c}] \} + \\
 & + \frac{Cp}{2} [3h' - 5cp C_b' h]
 \end{aligned} \tag{91}$$

$$a_{78} = 1 \tag{92}$$

$$a_{83} = \frac{s_c}{h} \rho P_{ro} \alpha R C_b' \tag{93}$$

$$a_{85} = \frac{1}{2h} \{ C_b' [\frac{1}{2} \frac{h'}{h} + C_b' (\frac{Cp}{2} + G)] - C_b'' \} \tag{94}$$

$$a_{86} = -C_b' / 2h \tag{95}$$

$$\begin{aligned}
 a_{87} = & \rho P_{ro} \alpha R P(w-c) \frac{s_c}{h + h'} + (\frac{Cp}{2} + G) (C_b'' + \frac{C_b'}{2} \frac{h'}{h}) - \\
 & C_b'^2 (\frac{3}{4} Cp^2 + G Cp + 2G^2)
 \end{aligned} \tag{96}$$

$$a_{88} = 2C_b' (C_p/2 + G) - \pi'/2 h \quad (97)$$

### Method of Solution

While the present report does not go beyond a formulation of the equation system it is possible to draw upon the work of Brown and Mack, references (8) and (9), and to develop a suggested method of solving the equation system, (60). In both of these references the authors constructed linearly independent solutions which were integrated across the boundary layer and were then combined to yield a general solution satisfying the boundary conditions. The two methods differed in the direction of integration and in the search procedure used to vary the initial guesses until a proper set of eigenvalues was obtained. The present suggested procedure follows more closely the method of Mack, reference (9).

Initially, we look for four linearly independent solutions which are applicable at  $\eta \geq \eta_0$ . These solutions are found by solving the equation system:

$$Z_i' = \sum_{j=1}^{\infty} a_{ij}^* Z_j \quad (i=1, \dots, 8) \quad (98)$$

where the superscript asterisk on the  $a_{ij}$  coefficients means that they take on their constant value at  $\eta \geq \eta_0$ . To facilitate the derivation of these solutions the dependent variables are changed by the substitutions:  $V_1 = Z$ ;  $V_2 = Z_4$ ;  $V_3 = Z_5$ ; and  $V_4 = Z_7$ . This changes the eight first order equations into four second order equations which are:

$$V_1'' = b_{11} V_1 + b_{12} V_2 + b_{13} V_3 + b_{14} V_4 \quad (99)$$

$$V_2'' = b_{22} V_2 + b_{23} V_3 + b_{24} V_4 \quad (100)$$

$$V_3'' = b_{32} V_2 + b_{33} V_3 + b_{34} V_4 \quad (101)$$

$$V_4'' = b_{44} V_4 \quad (102)$$

where the  $b_{ij}$ 's are the following combinations of the  $a_{ij}$ 's:

$$b_{11} = a_{12}^* a_{21}^* \quad (103)$$

$$b_{12} = a_{12}^* a_{24}^* \quad (104)$$

$$b_{13} = a_{12}^* a_{25}^* \quad (105)$$

$$b_{14} = a_{12}^* a_{27}^* \quad (106)$$

$$b_{22} = a_{42}^* a_{24}^* + a_{43}^* a_{34}^* a_{46}^* a_{64}^* \quad (107)$$

$$b_{23} = a_{42}^* a_{25}^* + a_{43}^* a_{35}^* + a_{46}^* a_{65}^* \quad (108)$$

$$b_{24} = a_{42}^* a_{27}^* + a_{43}^* a_{37}^* + a_{46}^* a_{67}^* \quad (109)$$

$$b_{32} = a_{56}^* a_{64}^* \quad (110)$$

$$b_{33} = a_{56}^* a_{65}^* \quad (111)$$

$$b_{34} = a_{56}^* a_{67}^* \quad (112)$$

$$b_{44} = a_{78}^* a_{87}^* \quad (113)$$

The system of equations (99) through (102) has particular solutions which may be written:

$$V_i^{(j)} = B_i^{(j)} e^{\lambda_j (\eta - \gamma_{15})} \quad \begin{matrix} i = 1, \dots, 4 \\ j = 1, \dots, 8 \end{matrix} \quad (114)$$

Putting the "j<sup>th</sup>" particular solution into the equations yield the following equations:

$$(b_{11} - \lambda_j^2) B_1^{(j)} + b_{12} B_2^{(j)} + b_{13} B_3^{(j)} + b_{14} B_4^{(j)} = 0 \quad (115)$$

$$(b_{22} - \lambda_j^2) B_2^{(j)} + b_{23} B_3^{(j)} + b_{24} B_4^{(j)} = 0 \quad (116)$$

$$b_{32} B_2^{(j)} + (b_{33} - \lambda_j^2) B_3^{(j)} + b_{34} B_4^{(j)} = 0 \quad (117)$$

$$(b_{44} - \lambda_j^2) B_4^{(j)} = 0 \quad (118)$$

The characteristic determinant yields four negative roots and four positive roots. The positive roots are discarded since they would give solutions increasing exponentially in  $\eta$ . The negative roots are:

$$\lambda_1 = - (b_{11})^{1/2} \quad (119)$$

$$\lambda_2 = - \left\{ \frac{1}{2} (b_{22} + b_{33}) + \left[ \frac{1}{4} (b_{22} - b_{33})^2 + b_{23} b_{32} \right]^{1/2} \right\}^{1/2} \quad (120)$$

$$\lambda_3 = - \left\{ \frac{1}{2} (b_{22} + b_{33}) - \left[ \frac{1}{4} (b_{22} - b_{33})^2 + b_{23} b_{32} \right]^{1/2} \right\}^{1/2} \quad (121)$$

$$\lambda_4 = - (b_{44})^{1/2} \quad (122)$$

Now by selection of a specific magnitude for one of the  $B_1^{(j)}$ 's in each of the four  $j$  groups of fundamental solutions the equations (115) to (118) can be solved to yield the following values:

For  $j = 1$ :

$$B_1^{(1)} = 1$$

$$B_2^{(1)} = 0$$

$$B_3^{(1)} = 0$$

$$B_4^{(1)} = 0$$

(123)

For  $j = 2$  or  $3$ :

$$B_1^{(j)} = \frac{b_{12}(b_{33} - \lambda_j^2) - b_{13}b_{32}}{\lambda_j^2 - b_{11}}$$

$$B_2^{(j)} = b_{33} - \lambda_j^2 \quad (124)$$

$$B_3^{(j)} = -b_{32}$$

$$B_4^{(j)} = 0$$

For  $j = 4$ :

$$B_1^{(4)} = \left[ -b_{14}[(b_{22} - b_{44})(b_{33} - b_{44}) - b_{23}b_{32}] - b_{12}[b_{24}(b_{44} - b_{33}) + b_{23}b_{34} + b_{13}[b_{34}(b_{22} - b_{44}) - b_{24}b_{32}]] \right] \cdot \left[ (b_{11} - b_{44}) \right. \\ \left. [(b_{22} - b_{44})(b_{33} - b_{44}) - b_{23}b_{32}] \right]^{-1} \quad (125)$$

$$B_2^{(4)} = \left\{ (b_{34}b_{23}) - b_{24}(b_{33} - b_{44}) \right\} \left\{ (b_{22} - b_{44})(b_{33} - b_{44}) - b_{23}b_{32} \right\}^{-1} \quad (126)$$

$$B_3^{(4)} = \left\{ b_{24}b_{32} - b_{34}(b_{22} - b_{44}) \right\} \left\{ (b_{22} - b_{44})(b_{33} - b_{44}) - b_{23}b_{32} \right\}^{-1} \quad (127)$$

$$B_4^{(4)} = 1 \quad (128)$$

Next, it is possible to write a general solution of equation (98) as:

$$Z_i = \sum_{j=1}^4 C_j A_i^{(j)} e^{\lambda_j(\eta - \eta_0)} \quad (i = 1, \dots, 8) \quad (129)$$

where we have used the fact that the four exponentially growing fundamental solutions are discarded. The  $A_i^{(j)}$ 's of equation (129) are now related to the  $B_i^{(j)}$ 's as follows:

$$A_1^{(j)} = B_1^{(j)} \quad (130)$$

$$A_2^{(j)} = \lambda_j B_1^{(j)} \quad (131)$$

$$A_3^{(j)} = \frac{i}{\lambda_j} \left\{ -B_1^{(j)} - \gamma_\infty M_\infty^2 (1-c) B_2^{(j)} + (1-c) B_3^{(j)} + \frac{1-c}{F(c_b)} B_4^{(j)} \right\} \quad (132)$$

$$A_4^{(j)} = B_2^{(j)} \quad (133)$$

$$A_5^{(j)} = B_3^{(j)} \quad (134)$$

$$A_6^{(j)} = \lambda_j B_3^{(j)} \quad (135)$$

$$A_7^{(j)} = B_4^{(j)} \quad (136)$$

$$A_8^{(j)} = \lambda_j B_4^{(j)} \quad (137)$$

The four  $j$  values now yield four fundamental solutions which may be numerically integrated from  $\eta = \eta_0$  to the wall,  $\eta = 0$ . At the wall the fundamental solutions are combined to give the general solutions which satisfy the remaining boundary conditions for high frequency fluctuations; namely,

$$z_1 = z_3 = z_5 = z_7 = 0 \quad (138)$$

CONCLUDING REMARKS

The present analysis extends the methods of direct solution of the boundary layer disturbances equations to account for the effects of foreign gas injection on the laminar boundary layer stability. The similarities between the present method and the methods of Brown and Mack, references (8) and (9), are obvious and do not require further comment. The major differences in the present method which result from the inclusion of mass transfer come from the addition of the species continuity equation, the diffusive flux terms in the energy equation, a modification of the form of the equation of state, and a difference in the dependence of the transport properties on the state variables. The present report covers only the formulation of the equation system, however, the analysis has been carried to a point where solutions may be sought by numerical methods.

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13 ABSTRACT The stability theory of laminar boundary layers in response to infinitesimal disturbances is re-examined for the case of a binary mixture with foreign gas injection. Because of the inherent limitations of the asymptotic stability calculation procedures, an approach was taken which utilizes the complete equation system. Such an approach is described herein and the resulting system of equations is presented in a manner suitable for numerical evaluation.		

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